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ScienceDirect
Journal of Sound and Vibration 320 (2009) 339–364
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Improvement to the averaging method using the Jacobian elliptic function

T. Okabe[a,*], T. Kondou[b]

[a]Department of Mechanical Systems Engineering, Faculty of Engineering, Miyazaki University, 1-1, Gakuen Kibanadai Nishi, Miyazaki 889-2192, Japan
[b]Department of Intelligent Machinery and Systems, Graduate School of Engineering, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 810-0315, Japan

Received 29 October 2007; received in revised form 11 March 2008; accepted 20 July 2008
Handling Editor: L.G. Tham
Available online 5 September 2008

Abstract

The averaging method was improved by using the Jacobian elliptic sine (sn), cosine (cn) and delta (dn) functions as generating solutions in order to obtain a highly accurate periodic solution for a strongly nonlinear dynamical system. The proposed method can be applied to a relatively general nonlinear system based on the single degree-of-freedom Duffing equation. Two methods of stability analysis were applied and evaluated for the approximate solutions obtained by the proposed method. The approximate solutions obtained by the proposed method for practical examples were compared to the solutions obtained by the shooting method. The results confirmed that the proposed method provides a more accurate solution and more accurate stability analysis results than those obtained by the conventional averaging method that uses trigonometric functions as the generating solution.

1. Introduction

The averaging method originally proposed by Krylov–Bogoliubov [1] is commonly used in the theoretical analysis of weak nonlinear dynamical systems because it has several methodological advantages. These advantages include a relatively straightforward computation procedure that analytically obtains both the periodic and transient solutions. This method can also be applied to a comparatively wide variety of nonlinear dynamical systems. The conventional averaging method, however, approximates the solution using only one term of trigonometric function, so the accuracy of the approximate solution for strongly nonlinear dynamical systems is poor. This is because the periodic solution of a nonlinear dynamical system contains many harmonic components, and the influence of the higher order harmonic components increases as the nonlinearity of the system becomes large. Therefore, it is important to investigate methods for improving the

*Corresponding author. Tel.: +81 985 58 7302; fax: +81 985 58 2876.
E-mail address: okabe@cc.miyazaki-u.ac.jp (T. Okabe).

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doi:10.1016/j.jsv.2008.07.014

analytical accuracy of the averaging method, even for strongly nonlinear dynamical systems, without diminishing the method's advantages.

Several methods have previously been proposed to improve the performance of the averaging method. These methods can be classified into two groups based on a methodological point of view. The first group includes secondary accuracy improvement methods [2] in which the higher harmonic components are incrementally added to the first-order approximate solution. These types of methods, however, are impractical because the computation procedures are very complicated.

The other group contains methods in which a more appropriate function, rather than a trigonometric function, is utilized as the generating solution. For this purpose, the Jacobian elliptic function was typically used. It is known that the Jacobian elliptic function is the exact solution for the undamped free vibration of a Duffing equation and simple pendulum. There are several papers that improve the averaging method by employing the Jacobian elliptic function as the generating solution for the following systems based on the Duffing equation:

$$\ddot{x} + \beta_1 x + \beta_3 x^3 = \varepsilon f, \tag{1}$$

and simple pendulum:

$$\ddot{x} + p^2 \sin x = \varepsilon f, \tag{2}$$

where β_1, β_3 and p are constants, ε is a small constant, and $f = f_1(x, \dot{x})$ or $f = f_2(x, \dot{x}, t)$ is the perturbed function. It is believed that the Jacobian elliptic function was first utilized as the generating solution of the averaging method by Barkham and Soudack [3]. They proposed the improved averaging method to solve an autonomous system in the case of $f = f_1(x, \dot{x})$ with $\beta_1 > 0$ and $\beta_3 > 0$ in Eq. (1). Christopher [4] developed a more accurate averaging method that employs the Jacobian elliptic function only for the case of $f(x, \dot{x}) = \dot{x}$ and $\beta_1, \beta_3 - (\varepsilon/2)^2 > 0$. Christopher and Brocklehurst [5] improved Christopher's method [4] and extended the range of applicability of this method. Yuste and Bejarano [6,7] showed that Christopher's method [4] can be extended to the oscillator of Eq. (1) with $\beta_1 > 0$, $\beta_3 < 0$ and $\beta_1 < 0$, $\beta_3 > 0$. Furthermore, Yusute and Bejarano [8] introduced a general expression for the time derivative of the amplitude and phase similar to those obtained by the conventional averaging method using a trigonometric function. Cap [9] devised an averaging method using the Jacobian elliptic function for the pendulum-type oscillator defined by Eq. (2) that incorporates various damping elements. Coppola and Rand [10] applied the averaging method using the Jacobian elliptic function to the oscillator of Eq. (1) with $f = f_1(x, \dot{x})$, and obtained the approximate solution of the limit cycle.

On the other hand, Roy [11] proposed an improved averaging method for a nonautonomous system defined by Eqs. (1) and (2) with $f = f_2(x, \dot{x}, t)$. Roy developed the averaging method using the Jacobian elliptic function to predict the steady-state solution in a harmonically excited strong nonlinear system. Hereafter, this method is referred to as Roy's method.

All of the above methods utilize the Jacobian elliptic function that expresses the exact solution of free vibration for the systems defined by Eqs. (1) and (2) with $\varepsilon = 0$. In the case in which the Jacobian elliptic function is used, the value of the modulus of elliptic integral must be determined. The modulus of elliptic integral is the most basic and important parameter, exerting a significant influence on the accuracy of the approximate solutions. In order to determine the value of the modulus, all of the previously mentioned methods utilized the relationship among amplitude, frequency and modulus that holds only in the case of undamped free vibration of Eqs. (1) and (2) with $\varepsilon = 0$ (see Section 2.3). However, the computational accuracy cannot be markedly improved when this relationship is utilized. This is verified numerically in Section 5 by comparing the computation accuracy of Roy's method that utilizes this relationship with the very accurate numerical solution obtained by the shooting method.

This paper introduces a new type of averaging method that uses the Jacobian elliptic function to obtain highly accurate periodic solutions for strongly nonlinear oscillators based on a single degree-of-freedom Duffing equation. In the proposed method, the Jacobian elliptic function is also used as the generating solution, but the relationship obtained from undamped free vibration is not used to determine the modulus of elliptic integral. Instead, a "*pseudo-averaged* equation" that incorporates the modulus as an unknown variable is derived by a skillful computational procedure and an accurate approximate solution is obtained from the equation.

The newly proposed method is collectively referred to as the averaging method of elliptic type, and the methods using the Jacobian elliptic cosine (cn), sine (sn) and delta (dn) functions are referred to as the averaging method of cn, sn and dn types, respectively. In addition, two types of stability analysis for the approximate solution obtained using the proposed method are also applied. The method's validity is verified by comparing the results obtained by applying the proposed method to typical nonlinear oscillators with the very accurate numerical solutions computed by the shooting method [12].

2. Preparative discussions for formulation of the method

2.1. Fundamental equation

Consider the single degree-of-freedom system governed by the following dimensionless differential equation with strong nonlinearity:

$$\left.\begin{aligned} &\ddot{x}+\beta_1 x+\beta_3 x^3=\varepsilon f(\omega t,x,\dot{x}),\\ &f(\omega t,x,\dot{x})=f(\omega t+2\pi,x,\dot{x}),\quad \beta_1=\pm 1,\quad \beta_3=\pm 1,\quad \cdot=\frac{\mathrm{d}}{\mathrm{d}t}. \end{aligned}\right\} \tag{3}$$

In mechanical systems, x,ω and t represent the displacement, angular frequency of excitation and time, respectively. $\varepsilon(0<\varepsilon\ll 1)$ is a small parameter that expresses the magnitude of perturbed function $f(\omega t,x,\dot{x})$, which is a nonlinear function of x and $\dot{x}$ and a periodic function with period 2π with respect to ωt. When $\varepsilon=0$, Eq. (3) reduces to the undamped free vibration response of the Duffing equation. The Duffing equation is a typical dynamic system that exhibits a strong and continuous nonlinearity. The exact solution to this equation can be expressed by the Jacobian elliptic function. The Duffing equation is categorized into three types depending on the combination of parameters β_1 and β_3. The groups are the hardening spring type when $(\beta_1,\beta_3)=(1,1)$, the softening spring type when $(\beta_1,\beta_3)=(1,-1)$ and the snap-through spring type when $(\beta_1,\beta_3)=(-1,1)$. This report addresses all three Duffing equation types.

In the following discussion, the highly accurate approximate solution for Eq. (3) with period 2π will be calculated using a Jacobian elliptic function as the generating solution for the averaging method.

2.2. Differential formulae for the Jacobian elliptic function

The Jacobian elliptic functions $\mathrm{sn}(v,k)$, $\mathrm{cn}(v,k)$ and $\mathrm{dn}(v,k)$ are collectively expressed by $\mathrm{ep}(v,k)$ in the following discussion, where $k(0\leqslant k<1)$ is the modulus of the Jacobian elliptic functions and integrals. The partial differentiations of the Jacobian elliptic function with respect to v are given by

$$\frac{\partial \mathrm{ep}(v,k)}{\partial v}=\mathrm{ep}'(v,k)=\begin{cases} \mathrm{sn}'(v,k)=\mathrm{cn}(v,k)\mathrm{dn}(v,k),\\ \mathrm{cn}'(v,k)=-\mathrm{sn}(v,k)\mathrm{dn}(v,k),\\ \mathrm{dn}'(v,k)=-k^2\mathrm{sn}(v,k)\mathrm{cn}(v,k), \end{cases} \tag{4}$$

where "$'$" $=\partial/\partial v$. In addition, the second partial differentiation of $\mathrm{ep}(v,k)$ with respect to v can be expressed as

$$\frac{\partial^2 \mathrm{ep}(v,k)}{\partial v^2}=\mathrm{ep}''(v,k)=2\rho_1\mathrm{ep}^3(v,k)+\rho_2\mathrm{ep}(v,k), \tag{5}$$

where $\rho_1=\rho_1(k)$ and $\rho_2=\rho_2(k)$ are both solely functions of modulus k and depend upon the type of Jacobian elliptic functions $\mathrm{ep}(v,k)$ used. These values are provided in Table 1.

On the other hand, the partial differentiation of $\mathrm{ep}(v,k)$ with respect to the modulus k can be obtained from the differential formulae [15] as

$$\frac{\partial \mathrm{ep}(v,k)}{\partial k}=\frac{1}{kl^2}\left[\left\{\left(l^2-\frac{E}{K}\right)v-Z(v,k)\right\}\mathrm{ep}'(v,k)+\rho_1\{1-\mathrm{ep}^2(v,k)\}\mathrm{ep}(v,k)\right], \tag{6}$$

Table 1
Definition of $\mathrm{ep}'(v,k)$, $\rho_1(k)$, $\rho_2(k)$ and $\rho_3(k)$ depending on $\mathrm{ep}(v,k)$

$\mathrm{ep}(v,k)$	$\mathrm{ep}'(v,k)$	$\rho_1(k)$	$\rho_2(k)$	$\rho_3(k)$
$\mathrm{sn}(v,k)$	$\mathrm{cn}(v,k)\mathrm{dn}(v,k)$	k^2	$-(k^2+1)$	0
$\mathrm{cn}(v,k)$	$-\mathrm{sn}(v,k)\mathrm{dn}(v,k)$	$-k^2$	$2k^2-1$	k^2
$\mathrm{dn}(v,k)$	$-k^2\,\mathrm{sn}(v,k)\mathrm{cn}(v,k)$	-1	$-(k^2-2)$	k^2

Table 2
Exact solution of the undamped free vibration of the Duffing equation

β_1	β_3	Type of spring property		Exact solution	n
1	1	Hardening spring system		$\hat{A}\,\mathrm{cn}(\hat{v},\hat{k})$	2
1	−1	Softening spring system		$\hat{A}\,\mathrm{sn}(\hat{v},\hat{k})$	2
−1	1	Snap-through spring system	full-swing mode	$\hat{A}\,\mathrm{cn}(\hat{v},\hat{k})$	2
			half-swing mode	$\hat{A}\,\mathrm{dn}(\hat{v},\hat{k})$	1

where $l=\sqrt{1-k^2}$ is the complementary modulus, $Z(v,\,k)$ the Jacobian zeta function, and $K=K(k)$ and $E=E(k)$ the complete elliptic integrals of the first and second kinds, respectively. In addition, the partial differentiation of $\mathrm{ep}'(v,\,k)$ with respect to k can be derived from Eqs. (4)–(6) as

$$\frac{\partial\,\mathrm{ep}'(v,k)}{\partial k}=\frac{1}{kl^2}\left[\left\{\left(l^2-\frac{E}{K}\right)v-Z(v,k)\right\}\{2\rho_1\,\mathrm{ep}^3(v,k)+\rho_2\,\mathrm{ep}(v,k)\}\right.$$
$$\left.-\{2\rho_1\,\mathrm{ep}^2(v,k)+\rho_3\}\mathrm{ep}'(v,k)\right]. \tag{7}$$

The definition of $\rho_3=\rho_3(k)$ in this equation is also provided in Table 1.

2.3. Exact solution of the undamped free vibration of the Duffing equation

When $\varepsilon=0$, Eq. (3) becomes

$$\ddot{x}+\beta_1 x+\beta_3 x^3=0. \tag{8}$$

Eq. (8) represents the undamped free vibration of the Duffing equation. The exact solution can be obtained analytically using the Jacobian elliptic functions [13,14], as shown in Table 2, except for the $(\beta_1,\beta_3)=(-1,-1)$ case, which has no vibrating solution. The $(\beta_1,\beta_3)=(-1,1)$ case, which is the snap-through spring system case, contains two types of exact solutions. These solutions are based on the three spring equilibrium points, $x=0$ (unstable) and $x=\pm1$ (stable). The symmetrical waveform solution given by the cn function is called the full-swing mode because the three equilibrium points all fall within the range of vibration. The asymmetrical waveform solution given by the dn function is called the half-swing mode because it vibrates around only one of the stable equilibrium points, $x=\pm1$.

The exact solution of Eq. (8) can be expressed using the following collective Jacobian elliptic function expression:

$$\hat{x}(t)=\hat{A}\,\mathrm{ep}(\hat{v},\hat{k}),\quad \hat{v}=\hat{\alpha}(\hat{\omega}t+\hat{\theta}),\quad \hat{\alpha}=\hat{\alpha}(\hat{k})=\frac{nK(\hat{k})}{\pi}. \tag{9}$$

In this equation, $\hat{A}$, $\hat{\omega}$, $\hat{\theta}$ and $\hat{k}$ are the amplitude, the nonlinear natural angular frequency, the phase angle and the modulus, respectively. The number n is equal to either 1 or 2, depending on the type of Jacobian

elliptic function used, as shown in Table 2. The phase angle $\hat{\theta}$ can be regarded as an arbitrary real number without a loss of generality.

The following equation is derived by substituting $\hat{x}(t)$ of Eq. (9) into x of Eq. (8) and combining the result with the relationship in Eq. (5):

$$(\hat{\alpha}^2\hat{\omega}^2\rho_2+\beta_1)\hat{A}\,\mathrm{ep}(\hat{v},\hat{k})+(2\hat{\alpha}^2\hat{\omega}^2\rho_1+\beta_3\hat{A}^2)\hat{A}\,\mathrm{ep}^3(\hat{v},\hat{k})=0. \tag{10}$$

The coefficients $\mathrm{ep}(\hat{v},\hat{k})$ and $\mathrm{ep}^3(\hat{v},\hat{k})$ are then set to zero. This gives the equations that determine the relationship between $\hat{A}$, $\hat{\omega}$ and $\hat{k}$ as follows:

$$\hat{\alpha}^2\hat{\omega}^2\hat{\rho}_2+\beta_1=0,\quad 2\hat{\alpha}^2\hat{\omega}^2\hat{\rho}_1+\beta_3\hat{A}^2=0. \tag{11}$$

3. Averaging method of elliptic type

3.1. Approximate solution assumption for a perturbed system

In order to derive a highly accurate approximate solution for the perturbed system described by Eq. (3), the Jacobian elliptic functions that correspond to the exact solution of the unperturbed system described by Eq. (8) are employed as the generating solution of the averaging method. As a result, Eq. (9) can be used to derive the approximate solution for the perturbed system as follows:

$$x(t)=A\,\mathrm{ep}(v,k),\quad \dot{x}(t)=A\alpha\omega\,\mathrm{ep}'(v,k), \tag{12}$$

where

$$v=\alpha u,\quad u=\omega t+\theta,\quad \alpha=\alpha(k)=\frac{nK(k)}{\pi}. \tag{13}$$

In these equations, the unknown variables are the amplitude $A=A(t)$, the phase angle $\theta=\theta(t)$ and the modulus $k=k(t)$, which are all assumed to be slowly varying functions with respect to time t. In steady-state vibration, A,θ and k become constants. In this case, the period of the steady-state vibration becomes $2\pi/\omega$ with respect to t, 2π with respect to u and $4K$ with respect to v.

In order to simplify the equations, the Jacobian elliptic function $\mathrm{ep}(v,k)$ and the zeta function $Z(v,k)$ are hereafter represented by ep and Z, respectively.

3.2. Differential operator with respect to time

Since the differentiation of K with respect to k is given by

$$\frac{\mathrm{d}K}{\mathrm{d}k}=\frac{1}{k}\left(\frac{E}{l^2}-K\right), \tag{14}$$

the differentiation of $v=\alpha u=nK(\omega t+\theta)/\pi$ with respect to t becomes

$$\dot{v}=\frac{n}{\pi}\left\{\frac{\dot{k}}{k}\left(\frac{E}{l^2}-K\right)u+K(\omega+\dot{\theta})\right\}. \tag{15}$$

The following differential operator with respect to time t is derived using Eq. (15) and by considering the fact that A, θ and k are all the functions of t:

$$\frac{\mathrm{d}}{\mathrm{d}t}=\dot{A}\frac{\partial}{\partial A}+\dot{k}\frac{\partial}{\partial k}+\frac{n}{\pi}\left\{\frac{\dot{k}}{k}\left(\frac{E}{l^2}-K\right)u+K(\omega+\dot{\theta})\right\}\frac{\partial}{\partial v}. \tag{16}$$

3.3. Derivation of the pseudo-averaged equations

Differentiating the first equation of Eq. (12) with respect to t by applying Eqs. (6) and (16) yields

$$\dot{x} = \dot{A}\,\mathrm{ep} + A\alpha(\omega + \dot{\theta})\mathrm{ep}' + \dot{k}\frac{A}{kl^2}\varphi_1, \tag{17}$$

where

$$\varphi_1 = -Z\,\mathrm{ep}' + \rho_1(\mathrm{ep} - \mathrm{ep}^3). \tag{18}$$

Comparing the second equation of Eq. (12) with Eq. (17) leads to

$$\dot{A}\,\mathrm{ep} + \dot{\theta}A\alpha\,\mathrm{ep}' + \dot{k}\frac{A}{kl^2}\varphi_1 = 0. \tag{19}$$

Next, differentiating the second equation of Eq. (12) with respect to t by applying Eqs. (5), (7) and (16) yields

$$\ddot{x} = \dot{A}\alpha\omega\,\mathrm{ep}' + A\alpha^2\omega(\omega + \dot{\theta})\varphi_2 + \dot{k}\frac{A\alpha\omega}{kl^2}\varphi_3, \tag{20}$$

where

$$\left.\begin{aligned} \varphi_2 &= \mathrm{ep}(2\rho_1\,\mathrm{ep}^2 + \rho_2), \\ \varphi_3 &= \left(\frac{E}{K} - l^2 - \rho_3\right)\mathrm{ep}' - 2\rho_1\,\mathrm{ep}^2\mathrm{ep}' - Z\varphi_2. \end{aligned}\right\} \tag{21}$$

Substituting Eqs. (12) and (20) into Eq. (3) leads to

$$\dot{A}\alpha\omega\,\mathrm{ep}' + \dot{\theta}A\alpha^2\omega\varphi_2 + \dot{k}\frac{A\alpha\omega\varphi_3}{kl^2} = \varepsilon f(u - \theta,\ A\,\mathrm{ep},\ A\alpha\omega\mathrm{ep}') - \eta, \tag{22}$$

where

$$\eta = (\alpha^2\omega^2\rho_2 + \beta_1)A\,\mathrm{ep} + (2\alpha^2\omega^2\rho_1 + \beta_3A^2)A\,\mathrm{ep}^3. \tag{23}$$

As $\varepsilon \to 0$, the values of A and k approach those of $\hat{A}$ and $\hat{k}$, respectively, from Eq. (11). Therefore, when ε is small, the coefficients of $A\,\mathrm{ep}$ and $A\,\mathrm{ep}^3$ in Eq. (23) become small values. As a result, η defined by Eq. (23) and the right-hand side of Eq. (22) become small values. If the magnitudes of $|\dot{A}|$, $|\dot{\theta}|$ and $|\dot{k}|$ are all small, the left-hand side of Eq. (22) also becomes a small value. Therefore, in the following discussions, it is assumed that the magnitudes of $|\dot{A}|$, $|\dot{\theta}|$ and $|\dot{k}|$ are all small. Then, A, θ and k can be regarded as slowly varying functions with respect to time t and can be averaged over one period in the same way as the conventional averaging method.

The averaged equations with respect to A, θ and k must be derived from Eqs. (19) and (22). However, the fact that the number of unknown variables is larger than the number of equations creates a serious problem. To overcome this obstacle, the first three equations are obtained by eliminating $\dot{A}$, $\dot{\theta}$ and $\dot{k}$ from Eqs. (19) and (22) individually. These three equations are then averaged over one period (0–2π) with respect to u. This technique for deriving the averaged equations is the key point in the formulation of the averaging method of elliptic type.

During the time averaging procedure, unknown variables A, θ and k are assumed to be constants, since they are the slowly varying functions with respect to time. The property that allows the Jacobian elliptic functions, φ_1, φ_2 and φ_3 to be expanded into the Fourier series [15] as shown in Table 3 is also utilized.

Hereafter, the variables marked with "–" denote averaged values. Therefore, $\bar{K} = K(\bar{k})$, $\bar{E} = E(\bar{k})$, $\bar{\alpha} = \alpha(\bar{k}) = n\bar{K}/\pi$.

First, eliminating $\dot{A}$ from Eqs. (19) and (22) yields the following equation:

$$\dot{\theta}A\alpha^2(\varphi_2\,\mathrm{ep} - \mathrm{ep}'^2) + A\omega^2\alpha^2\varphi_2\,\mathrm{ep} + \beta_1A\,\mathrm{ep}^2 + \beta_3A^3\,\mathrm{ep}^4 = \varepsilon f(u - \bar{\theta}, \bar{A}\mathrm{ep}, \bar{A}\bar{\alpha}\omega\,\mathrm{ep}')\mathrm{ep} \tag{24}$$

Table 3
Fundamental forms of Fourier series of Jacobian elliptic functions, φ_1, φ_2 and φ_3

Functions	Fourier series
sn,sncn,sndn	$S_m \sin[(2m+1)nu/2]$
cn,cndn	$C_m \cos[(2m+1)nu/2]$
dn	$C_m \cos(mnu)$
Z	$S_m \sin[(m+1)u]$
φ_1,φ_2	$S_m \sin[(2m+1)nu/2]$
φ_3	$C_m \cos[(2m+1)nu/2]$ } ep = cn ∪ dn
φ_1,φ_2	$C_m \cos[(2m+1)nu/2]$
φ_3	$S_m \sin[(2m+1)nu/2]$ } ep = sn

C_m and $S_m (m = 0,1,2,\ldots)$ are mth order cosine and sine Fourier coefficient of the corresponding function. The definition of n is in Table 2.

By averaging Eq. (24) over one period with respect to u and by using the relationship shown in Table 3 and the orthogonality between cosine and sine functions, the following approximate averaged equation is derived:

$$\dot{\bar{\theta}}\bar{A}\omega\bar{\Psi}_1 + \bar{A}\omega^2\bar{\Psi}_2 + \beta_1\bar{A}\bar{\Psi}_3 + \beta_3\bar{A}^3\bar{\Psi}_4 = \varepsilon\bar{H}_1, \tag{25}$$

where

$$\left.\begin{aligned} \bar{\Psi}_1 = \Psi_1(\bar{k}) = \bar{\alpha}^2\int_0^{2\pi}(\varphi_2 \mathrm{ep} - \mathrm{ep}'^2)\mathrm{d}u, \quad \bar{\Psi}_2 = \Psi_2(\bar{k}) = \bar{\alpha}^2\int_0^{2\pi}\varphi_2\,\mathrm{ep}\,\mathrm{d}u, \\ \bar{\Psi}_3 = \Psi_3(\bar{k}) = \int_0^{2\pi}\mathrm{ep}^2\mathrm{d}u, \quad \bar{\Psi}_4 = \Psi_4(\bar{k}) = \int_0^{2\pi}\mathrm{ep}^4\,\mathrm{d}u, \\ \bar{H}_1 = H_1(\bar{A},\bar{\theta},\bar{k}) = \int_0^{2\pi} f(u-\bar{\theta}, \ \bar{A}\,\mathrm{ep}, \bar{A}\bar{\alpha}\omega\,\mathrm{ep}')\mathrm{ep}\,\mathrm{d}u. \end{aligned}\right\} \tag{26}$$

Next, the equation derived by eliminating $\dot{\theta}$ from Eqs. (19) and (22) is averaged, yielding

$$\dot{\bar{A}}\omega\bar{\Psi}_5 + \dot{\bar{k}}\,\bar{A}\omega\,\bar{\Psi}_6 = \varepsilon\bar{H}_2, \tag{27}$$

where

$$\left.\begin{aligned} \bar{\Psi}_5 = \Psi_5(\bar{k}) = \frac{1}{\bar{\alpha}}\bar{\Psi}_1, \quad \bar{\Psi}_6 = \Psi_6(\bar{k}) = \frac{\bar{\alpha}}{\bar{k}\bar{l}^2}\int_0^{2\pi}(\varphi_1\varphi_2 - \varphi_3\mathrm{ep}')\mathrm{d}u, \\ \bar{H}_2 = H_2(\bar{A},\bar{\theta},\bar{k}) = -\int_0^{2\pi} f(u-\bar{\theta}, \bar{A}\,\mathrm{ep}, \bar{A}\bar{\alpha}\omega\,\mathrm{ep}')\mathrm{ep}'\,\mathrm{d}u. \end{aligned}\right\} \tag{28}$$

Finally, the equation derived by eliminating $\dot{k}$ from Eqs. (19) and (22) is averaged, resulting in

$$\dot{\bar{\theta}}\bar{A}\omega\bar{\Psi}_7 + \bar{A}\omega^2\bar{\Psi}_8 + \beta_1\bar{A}\bar{\Psi}_9 + \beta_3\bar{A}^3\bar{\Psi}_{10} = \varepsilon\bar{H}_3, \tag{29}$$

where

$$\left.\begin{aligned} \bar{\Psi}_7 = \Psi_7(\bar{k}) = \bar{\alpha}\bar{k}\bar{l}^2\bar{\Psi}_6, \quad \bar{\Psi}_8 = \Psi_8(\bar{k}) = \bar{\alpha}^2\int_0^{2\pi}\varphi_1\varphi_2\,\mathrm{d}u, \\ \bar{\Psi}_9 = \Psi_9(\bar{k}) = \int_0^{2\pi}\varphi_1\,\mathrm{ep}\,\mathrm{d}u, \quad \bar{\Psi}_{10} = \bar{\Psi}_{10}(\bar{k}) = \int_0^{2\pi}\varphi_1\,\mathrm{ep}^3\,\mathrm{d}u, \\ \bar{H}_3 = H_3(\bar{A},\bar{\theta},\bar{k}) = \int_0^{2\pi} f(u-\bar{\theta}, \bar{A}\,\mathrm{ep}, \bar{A}\bar{\alpha}\omega\,\mathrm{ep}')\varphi_1\,\mathrm{d}u. \end{aligned}\right\} \tag{30}$$

The results for $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$ in Eqs. (25), (27) and (29) change depending on the form of the perturbed function $f(\omega t, x, \dot{x})$. On the other hand, Ψ_i $(i = 1,2,\ldots,10)$ are independent of $f(\omega t, x, \dot{x})$ and are only functions of $\bar{k}$. They can be calculated analytically depending on the type of elliptic function, as presented below.

(i) In the case of ep = cn:

$$\left.\begin{aligned}
&\bar{\Psi}_1=\frac{16\bar{K}^2}{3\pi\bar{k}^2}\left\{(1-2\bar{k}^2)\frac{\bar{E}}{\bar{K}}-\bar{l}^2\right\},\quad \bar{\Psi}_2=\frac{\bar{\Psi}_1}{2},\quad \bar{\Psi}_3=\frac{2\pi}{\bar{k}^2}\left(\frac{\bar{E}}{\bar{K}}-\bar{l}^2\right),\\
&\bar{\Psi}_4=\frac{2\pi}{3\bar{k}^4}\left\{2(2\bar{k}^2-1)\frac{\bar{E}}{\bar{K}}+(2-3\bar{k}^2)\bar{l}^2\right\},\quad \bar{\Psi}_5=\frac{1}{\bar{\alpha}}\bar{\Psi}_1,\quad \bar{\Psi}_6=\frac{4\bar{K}}{3\bar{k}^3\bar{l}^2}\left\{(1-2\bar{k}^2)\frac{\bar{E}^2}{\bar{K}^2}-4\bar{l}^2\frac{\bar{E}}{\bar{K}}+3\bar{l}^2\right\},\\
&\bar{\Psi}_7=\bar{\alpha}\bar{k}\,\bar{l}^2\bar{\Psi}_6,\quad \bar{\Psi}_8=\frac{\bar{\Psi}_7}{2},\quad \bar{\Psi}_9=-\frac{\pi}{\bar{k}^2}\left(\frac{\bar{E}^2}{\bar{K}^2}-\bar{l}^2\right),\quad \bar{\Psi}_{10}=\frac{\pi}{3\bar{k}^4}\left\{(1-2\bar{k}^2)\frac{\bar{E}^2}{\bar{K}^2}+2\bar{l}^2\frac{\bar{E}}{\bar{K}}-3\bar{l}^4\right\}.
\end{aligned}\right\} \tag{31}$$

(ii) In the case of ep = sn:

$$\left.\begin{aligned}
&\bar{\Psi}_1=\frac{16\bar{K}^2}{3\pi\bar{k}^2}\left\{-(1+\bar{k}^2)\frac{\bar{E}}{\bar{K}}+\bar{l}^2\right\},\quad \bar{\Psi}_2=\frac{\bar{\Psi}_1}{2},\quad \bar{\Psi}_3=-\frac{2\pi}{\bar{k}^2}\left(\frac{\bar{E}}{\bar{K}}-1\right),\\
&\bar{\Psi}_4=-\frac{2\pi}{3\bar{k}^4}\left\{2(1+\bar{k}^2)\frac{\bar{E}}{\bar{K}}-(2+\bar{k}^2)\right\},\quad \bar{\Psi}_5=\frac{1}{\bar{\alpha}}\bar{\Psi}_1,\\
&\bar{\Psi}_6=-\frac{4\bar{K}}{3\bar{k}^3\bar{l}^2}\left\{(1+\bar{k}^2)\frac{\bar{E}^2}{\bar{K}^2}-4\bar{l}^2\frac{\bar{E}}{\bar{K}}+3\bar{l}^4\right\},\\
&\bar{\Psi}_7=\bar{\alpha}\bar{k}\,\bar{l}^2\bar{\Psi}_6,\quad \bar{\Psi}_8=\frac{\bar{\Psi}_7}{2},\quad \bar{\Psi}_9=\frac{\pi}{\bar{k}^2}\left(\frac{\bar{E}^2}{\bar{K}^2}-\bar{l}^2\right),\\
&\bar{\Psi}_{10}=\frac{\pi}{3\bar{k}^4}\left\{(1+\bar{k}^2)\frac{\bar{E}^2}{\bar{K}^2}+2\bar{l}^2\frac{\bar{E}}{\bar{K}}-3\bar{l}^2\right\}.
\end{aligned}\right\} \tag{32}$$

(iii) In the case of ep = dn:

$$\left.\begin{aligned}
&\bar{\Psi}_1=\frac{4\bar{K}^2}{3\pi}\left\{(\bar{k}^2-2)\frac{\bar{E}}{\bar{K}}+2\bar{l}^2\right\},\quad \bar{\Psi}_2=\frac{\bar{\Psi}_1}{2},\quad \bar{\Psi}_3=2\pi\frac{\bar{E}}{\bar{K}},\\
&\bar{\Psi}_4=\frac{2\pi}{3}\left\{2(2-\bar{k}^2)\frac{\bar{E}}{\bar{K}}-\bar{l}^2\right\},\quad \bar{\Psi}_5=\frac{1}{\bar{\alpha}}\bar{\Psi}_1,\quad \bar{\Psi}_6=-\frac{2\bar{K}}{3\bar{k}\,\bar{l}^2}\left\{(2-\bar{k}^2)\frac{\bar{E}^2}{\bar{K}^2}-2\bar{l}^2(2+\bar{k}^2)\frac{\bar{E}}{\bar{K}}+\bar{l}^2(2+\bar{k}^2)\right\},\\
&\bar{\Psi}_7=\bar{\alpha}\bar{k}\,\bar{l}^2\bar{\Psi}_6,\quad \bar{\Psi}_8=\frac{\bar{\Psi}_7}{2},\quad \bar{\Psi}_9=-\pi\left\{\frac{\bar{E}^2}{\bar{K}^2}-2\bar{l}^2\frac{\bar{E}}{\bar{K}}+\bar{l}^2\right\},\quad \bar{\Psi}_{10}=\frac{\pi}{3}\left\{(\bar{k}^2-2)\frac{\bar{E}^2}{\bar{K}^2}+4\bar{l}^4\frac{\bar{E}}{\bar{K}}-2\bar{l}^4\right\}.
\end{aligned}\right\} \tag{33}$$

It should be noted that the averaged equations (25), (27) and (29) are derived approximately by applying the skillful procedure to Eqs. (19) and (22). Therefore, Eqs. (25), (27) and (29) are called "*pseudo-averaged equations*."

3.4. Computation of the periodic solution

The periodic solution of Eq. (3) can be obtained by setting $\dot{\bar{A}}=\dot{\bar{\theta}}=\dot{\bar{k}}=0$ in Eqs. (25), (27) and (29). That is, $\bar{A}$, $\bar{\theta}$ and $\bar{k}$ in the periodic solution are calculated as the roots of the following equations:

$$\mathbf{G}(\bar{\mathbf{y}})=\mathbf{0}, \tag{34}$$

where

$$\mathbf{G}=\begin{pmatrix} G_1(\bar{A},\bar{\theta},\bar{k}) \\ G_2(\bar{A},\bar{\theta},\bar{k}) \\ G_3(\bar{A},\bar{\theta},\bar{k}) \end{pmatrix}=\begin{pmatrix} \bar{A}\omega^2\bar{\Psi}_2+\beta_1\bar{A}\bar{\Psi}_3+\beta_3\bar{A}^3\bar{\Psi}_4-\varepsilon\bar{H}_1 \\ \varepsilon\bar{H}_2 \\ \bar{A}\omega^2\bar{\Psi}_8+\beta_1\bar{A}\bar{\Psi}_9+\beta_3\bar{A}^3\bar{\Psi}_{10}-\varepsilon\bar{H}_3 \end{pmatrix},\quad \bar{\mathbf{y}}=\begin{pmatrix} \bar{A} \\ \bar{\theta} \\ \bar{k} \end{pmatrix}. \tag{35}$$

Eq. (34) is a system of nonlinear algebraic equations with respect to $\bar{A}$, $\bar{\theta}$ and $\bar{k}$. When Eq. (34) is solved by the Newton–Raphson method, the procedure of successive approximation is expressed by

$$\bar{\mathbf{y}}_{n+1}=\bar{\mathbf{y}}_n-\left[\frac{\partial\mathbf{G}(\bar{\mathbf{y}}_n)}{\partial\bar{\mathbf{y}}}\right]^{-1}\mathbf{G}(\bar{\mathbf{y}}_n). \tag{36}$$

If $\det[\partial\mathbf{G}/\partial\bar{\mathbf{y}}]\neq 0$ in the neighborhood of the solution of Eq. (34), Eq. (36) can be solved successfully. It has been numerically verified that $\det[\partial\mathbf{G}/\partial\bar{\mathbf{y}}]$ is not equal to zero for many typical nonlinear oscillators.

3.5. Relationship with the conventional averaging method

In this section, we consider the case where the generating solution of the averaging method is assumed to be cn or sn. When $k=0$, the following relationships hold:

$$\left.\begin{aligned} \mathrm{cn}(u,0)=\cos u,\ \mathrm{sn}(u,0)=\sin u,\ \mathrm{dn}(u,0)=1, \\ Z(u,0)=0,\ K(0)=E(0)=\pi/2. \end{aligned}\right\} \tag{37}$$

Therefore, as $k\to 0$, the generating solution approaches the following trigonometric functions:

$$\left.\begin{aligned} x=A\cos\omega t,\quad \dot{x}=-A\omega\sin\omega t\ (\text{for ep}=\text{cn}), \\ x=A\sin\omega t,\quad \dot{x}=A\omega\cos\omega t\ (\text{for ep}=\text{sn}), \end{aligned}\right\} \tag{38}$$

Eq. (38) is identical to the generating solution assumed by the conventional averaging method. K, E and E/K can be expanded into power series with respect to k as

$$\left.\begin{aligned} K=\frac{\pi}{2}\left\{1+\frac{1}{4}k^2+\frac{9}{64}k^4+o(k^6)\right\}, \\ E=\frac{\pi}{2}\left\{1-\frac{1}{4}k^2-\frac{3}{64}k^4+o(k^6)\right\}, \\ \frac{E}{K}=1-\frac{1}{2}k^2-\frac{1}{16}k^4+o(k^6). \end{aligned}\right\} \tag{39}$$

When the limit of $k\ (=\bar{k})\to 0$ is applied to Eqs. (38) and (39), the first and second row of Eq. (34) become

$$\left.\begin{aligned} (\beta_1-\omega^2)\bar{A}+\frac{3}{4}\beta_3\bar{A}^3=\frac{\varepsilon}{\pi}\int_0^{2\pi}f(u-\bar{\theta},\bar{A}\cos u,-\bar{A}\omega\sin u)\cos u\,\mathrm{d}u, \\ \int_0^{2\pi}f(u-\bar{\theta},\bar{A}\cos u,-\bar{A}\omega\sin u)\sin u\,\mathrm{d}u=0 \end{aligned}\right\}\quad (\text{for ep}=\text{cn}), \tag{40}$$

$$\left.\begin{aligned} (\beta_1-\omega^2)\bar{A}+\frac{3}{4}\beta_3\bar{A}^3=\frac{\varepsilon}{\pi}\int_0^{2\pi}f(u-\bar{\theta},\bar{A}\sin u,\ \bar{A}\omega\cos u)\sin u\,\mathrm{d}u, \\ \int_0^{2\pi}f(u-\bar{\theta},\ \bar{A}\sin u,\ \bar{A}\omega\cos u)\cos u\,\mathrm{d}u=0 \end{aligned}\right\}\quad (\text{for ep}=\text{sn}). \tag{41}$$

Eqs. (40) and (41) are indistinguishable from the equations used by the conventional averaging method to determine the approximate solution. And since $\bar{\Psi}_8$, $\bar{\Psi}_9$, $\bar{\Psi}_{10}$ and $\bar{H}_3\to 0$ as $\bar{k}\to 0$, the third row of Eq. (34)

always holds at $\bar{k}=0$. Therefore, that the averaging method of cn and sn types are rational generalizations of the conventional averaging method.

3.6. Ineffective application

In the next case, the perturbed function $f(\omega t, x, \dot{x})$ contains even powers with respect to x.

$$f(x) = x^{2n} \quad (n = 1, 2, 3, \ldots). \tag{42}$$

When ep = cn or ep = sn, $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$ become

$$\left.\begin{aligned} \bar{H}_1 &= \bar{A}^{2n}\int_0^{2\pi} \mathrm{ep}^{2n+1}\,\mathrm{d}u = 0, \\ \bar{H}_2 &= -\bar{A}^{2n}\int_0^{2\pi} \mathrm{ep}^{2n}\mathrm{ep}'\,\mathrm{d}u = 0, \\ \bar{H}_3 &= \bar{A}^{2n}\int_0^{2\pi} \mathrm{ep}^{2n}\{-Z\,\mathrm{ep}' + \rho_1(\mathrm{ep} - \mathrm{ep}^3)\}\mathrm{d}u = 0. \end{aligned}\right\} \tag{43}$$

This result indicates that the even ordered terms x^{2n} do not affect the approximate solution. This can be attributed to the unconformity between the exact solution and the cn or sn generating solutions. That is, the exact solution can be expressed by a Fourier series containing both even and odd orders, but the Fourier series for the cn and sn contain only odd order terms. Therefore, the accuracy of the averaging method of cn and sn types cannot be improved for systems containing even order terms x^{2n}. When a spring function of a system has a quadratic term x^2 and this term exerts a serious influence on its vibration characteristics, the exact solution $x = B - A\,\mathrm{sn}^2$ (A and B: constants) of undamped free vibration for the quadratic oscillator $\ddot{x} + 1 + x^2 = 0$ can be used as the generation solution of elliptic averaging method. This result will be detailed in a subsequent report.

4. Stability analysis

Because the averaging method of elliptic type obtains both stable and unable solutions, it is necessary to evaluate their stability. In this section, two methods of stability analysis are applied to the approximate solutions obtained from the proposed method.

4.1. Method using the variational equation of the fundamental equation

The variational equation of Eq. (3) is expressed in the normal form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta_1 - 3\beta_3 x^2 + \varepsilon\dfrac{\partial f}{\partial x} & \varepsilon\dfrac{\partial f}{\partial \dot{x}}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}. \tag{44}$$

In this equation, $\xi_1 = \xi$ and $\xi_2 = \dot{\xi}$, where ξ denotes a small variation of x. When x is a periodic solution, Eq. (44) becomes a system of linear ordinary differential equations with periodic coefficients. The stability of the zero solution of Eq. (44), and consequently, the stability of the periodic solution of Eq. (3) can be determined from the two eigenvalues of the state transition matrix, i.e., the characteristic multipliers of Eq. (44). If the absolute values of the two characteristic multipliers are both less than one, then the periodic solution x is stable. If one or more of the characteristic multipliers are greater than one, then the solution is unstable.

This type of method, which is based on Floquet's theorem, is typically used for the stability analysis of periodic solutions in nonlinear systems because of the very high accuracy in evaluating the stability. On the other hand, a numerical integration technique, such as the Runge-Kutta-Gill (RKG) method, must be employed to compute the state transition matrix because it cannot be obtained analytically. Therefore, this method requires a relatively large computational effort.

4.2. Method using the variational equation of the averaging equation

Denoting small variations of $\bar{A}$, $\bar{\theta}$ and $\bar{k}$ by δA, $\delta\theta$ and δk, the variational equations of averaging equations (25), (27) and (29) with respect to $\bar{A}$, $\bar{\theta}$ and $\bar{k}$ are provided as

$$\left.\begin{aligned} \omega\bar{A}\bar{\Psi}_1\delta\dot{\theta} &= \bar{X}_{11}\delta A + \bar{X}_{12}\delta\theta + \bar{X}_{13}\delta k,\\ \omega\bar{\Psi}_5\delta\dot{A} + \omega\bar{A}\bar{\Psi}_6\delta\dot{k} &= \bar{X}_{21}\delta A + \bar{X}_{22}\delta\theta + \bar{X}_{23}\delta k,\\ \omega\bar{A}\bar{\Psi}_7\delta\dot{\theta} &= \bar{X}_{31}\delta A + \bar{X}_{32}\delta\theta + \bar{X}_{33}\delta k, \end{aligned}\right\} \tag{45}$$

where

$$\left.\begin{aligned} \bar{X}_{11} &= \varepsilon\frac{\partial\bar{H}_1}{\partial\bar{A}} - \omega^2\bar{\Psi}_2 - \beta_1\bar{\Psi}_3 - 3\beta_3\bar{A}^2\bar{\Psi}_4, \quad \bar{X}_{12} = \varepsilon\frac{\partial\bar{H}_1}{\partial\bar{\theta}},\\ \bar{X}_{13} &= \varepsilon\frac{\partial\bar{H}_1}{\partial\bar{k}} - \omega^2\bar{A}\frac{\mathrm{d}\bar{\Psi}_2}{\mathrm{d}\bar{k}} - \beta_1\bar{A}\frac{\mathrm{d}\bar{\Psi}_3}{\mathrm{d}\bar{k}} - \beta_3\bar{A}^3\frac{\mathrm{d}\bar{\Psi}_4}{\mathrm{d}\bar{k}}, \quad \bar{X}_{21} = \varepsilon\frac{\partial\bar{H}_2}{\partial\bar{A}}, \quad \bar{X}_{22} = \varepsilon\frac{\partial\bar{H}_2}{\partial\bar{\theta}},\\ \bar{X}_{23} &= \varepsilon\frac{\partial\bar{H}_2}{\partial\bar{k}}, \quad \bar{X}_{31} = \varepsilon\frac{\partial\bar{H}_3}{\partial\bar{A}} - \omega^2\bar{\Psi}_8 - \beta_1\bar{\Psi}_9 - 3\beta_3\bar{A}^2\bar{\Psi}_{10}, \quad \bar{X}_{32} = \varepsilon\frac{\partial\bar{H}_3}{\partial\bar{\theta}},\\ \bar{X}_{33} &= \varepsilon\frac{\partial\bar{H}_3}{\partial\bar{k}} - \omega^2\bar{A}\frac{\mathrm{d}\bar{\Psi}_8}{\mathrm{d}\bar{k}} - \beta_1\bar{A}\frac{\mathrm{d}\bar{\Psi}_9}{\mathrm{d}\bar{k}} - \beta_1\bar{A}^3\frac{\mathrm{d}\bar{\Psi}_{10}}{\mathrm{d}\bar{k}}. \end{aligned}\right\} \tag{46}$$

In this section, the symbols marked with "–" represent the values corresponding to the periodic solution. The solution of Eq. (45) is expressed using the constants $\delta\tilde{A}$, $\delta\tilde{\theta}$, $\delta\tilde{k}$ and the characteristic root λ, which dominates the stability of the solution, as follows:

$$\delta A = \delta\tilde{A}\,\mathrm{e}^{\lambda t}, \quad \delta\theta = \delta\tilde{\theta}\,\mathrm{e}^{\lambda t}, \quad \delta k = \delta\tilde{k}\,\mathrm{e}^{\lambda t}. \tag{47}$$

The characteristic root λ can be determined by substituting Eq. (47) into Eq. (45) and applying the existence condition for the nontrivial solutions of $\delta\tilde{A}$, $\delta\tilde{\theta}$ and $\delta\tilde{k}$:

$$\det\begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} - \lambda\omega\bar{A}\bar{\Psi}_1 & \bar{X}_{13}\\ \bar{X}_{21} - \lambda\omega\,\bar{\Psi}_5 & \bar{X}_{22} & \bar{X}_{23} - \lambda\omega\bar{A}\bar{\Psi}_6\\ \bar{X}_{31} & \bar{X}_{32} - \lambda\omega\bar{A}\bar{\Psi}_7 & \bar{X}_{33} \end{bmatrix} = 0 \tag{48}$$

Eq. (48) is a quadratic equation with respect to λ. The stability of the periodic solution can be evaluated by determining λ from Eq. (48). If the real parts of the two characteristic roots are both negative, then the periodic solution is stable. If at least one of them is positive, then it is unstable.

This type of method is easy to understand and requires only a small computational effort. For these reasons, it is frequently used for the stability analysis of periodic solutions obtained by the averaging method. On the other hand, the accuracy of the stability analysis is poor because this method may not obtain the secondary unstable regions, as shown in the following numerical examples.

5. Computational results and discussion

5.1. Numerical calculation conditions

The validity of the proposed method is verified by applying the method to several nonlinear oscillators. The accuracy of the computational results is then compared with that obtained by the shooting method [11] and the conventional averaging method. The numerical integral in the shooting method is performed by the RKG method and the step size for the RKG method is set to $2\pi/1024$. The numerical solution obtained by the shooting method is very accurate and can be regarded as the correct solution. The thick and the thin lines in the following figures represent the approximate solutions calculated by the proposed method (x) and the conventional averaging method (x_t), respectively. The solid and broken lines represent the stable and unstable solutions, respectively, and the symbols $\square$, $\triangle$ and $\triangledown$ denote the boundary between stable and unstable regions for saddle-node, pitchfork and Hopf bifurcations, respectively. The accurate numerical solutions x_s obtained

by the shooting method are depicted using ○ for a stable solution and ● for an unstable solution, in addition to the thin lines. The dot-dashed line represents the exact solution of undamped free vibration.

When the perturbed function $f(\omega t, x, \dot{x})$ can be defined by a simple function, $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$ in Eq. (25)–(30) can be calculated analytically using the Fourier series expansion of the Jacobian elliptic function, called the q-expansion [15]. However, it is difficult to calculate $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$ analytically for arbitrary $f(\omega t, x, \dot{x})$. Instead, numerical integration is employed. $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$ in the following calculations are computed using Simpson's rule, a well-known numerical integration method. In addition, it is also difficult to calculate the Jacobian matrix elements that are needed to solve Eq. (34) using the Newton–Raphson method and the partial derivatives with respect to $\bar{A}$, $\bar{\theta}$ and $\bar{k}$ in the determinant of Eq. (46) analytically. Therefore, a numerical difference is employed for these calculations. By adopting these numerical calculation techniques, a program with a wide range of applicability can be developed.

The stability of the periodic solution obtained by the proposed method is determined using the stability criterion defined by the variational equation of the fundamental equation discussed in Section 4.1. The stability of the periodic solution obtained by the conventional averaging method is determined from a similar procedure described in Section 4.2, which determines the behavior of the amplitude and phase angle after small disturbances.

5.2. Averaging method of cn type for a Duffing oscillator with hardening spring

Although the proposed method can be applied to a variety of systems containing a complex perturbed function, this section applied the method to a Duffing equation with a hard spring ($\beta_1 = \beta_3 = 1$) in order to verify the validity of the proposed method in detail for the most fundamental model of a nonlinear system. The perturbed function $f(\omega t, x, \dot{x})$ of a forced and damped Duffing oscillator can be generally expressed as

$$f(\omega t, x, \dot{x}) = F \cos \omega t - c\dot{x}, \tag{49}$$

where F and ω are the amplitude and angular frequency of the external force, respectively, and c is the viscous damping coefficient. The parameters in Eq. (49) were set to $F = 1.0$, $c = 0.02$ and $\varepsilon = 1.0$. The steady-state solution for a Duffing oscillator with hardening spring is calculated using the averaging method of cn type.

The frequency responses of the amplitude are shown in Fig. 1 in order to display the overall frequency response characteristics and computational accuracy of the approximate solutions obtained by the proposed and conventional averaging methods. The figure reveals that the main resonance of the approximate solutions obtained by the proposed method coincides with the numerical solutions obtained by the shooting method. On the other hand, the frequency response curves obtained by the conventional averaging method are quantitatively different from those for the shooting method.

This paper examines the characteristics of the two types of stability analysis discussed in Section 4, that is, one is the method by using the variational equation of the fundamental equation (method I) and the other by using the variational equation of the averaging equation (method II). The vertical tangents of the frequency response curves can be determined by both stability criteria as the boundaries between the stable and unstable regions caused by saddle-node bifurcations. Fig. 2 provides a detailed view of the frequency response curve in the low frequency region ($\omega = 0.35$–0.8), which is the left flank of the main resonance. The pitchfork bifurcation point calculated by the shooting method is located at $(\omega_p,\ x_p) = (0.6905, 0.9523)$ and $(\omega_p,\ x_p) = (0.7407, 0.9716)$. The solution becomes locally unstable in the region between the two pitchfork bifurcation points and the other solution which represents the higher harmonic resonance of second-order bifurcates and generate from these points. When the method I is employed for the averaging method of cn type, two pitchfork bifurcation points appear at $(\omega_p, x_p) = (0.6906, 0.9536)$ and $(\omega_p, x_p) = (0.7404, 0.9723)$. The locations of the two boundary points and stability characteristics obtained by the method I nearly correspond with those determined by the shooting method, both qualitatively and quantitatively. On the other hand, when the method II is employed, the approximate solutions for the entire region in the left flank of the main resonance are judged to be stable. This result means that the method II cannot accurately determine the stability of the previously mentioned secondary unstable region. The accuracy of stability analysis is strongly dependent upon the accuracy of the approximate solution. Because the averaging method of cn type provides

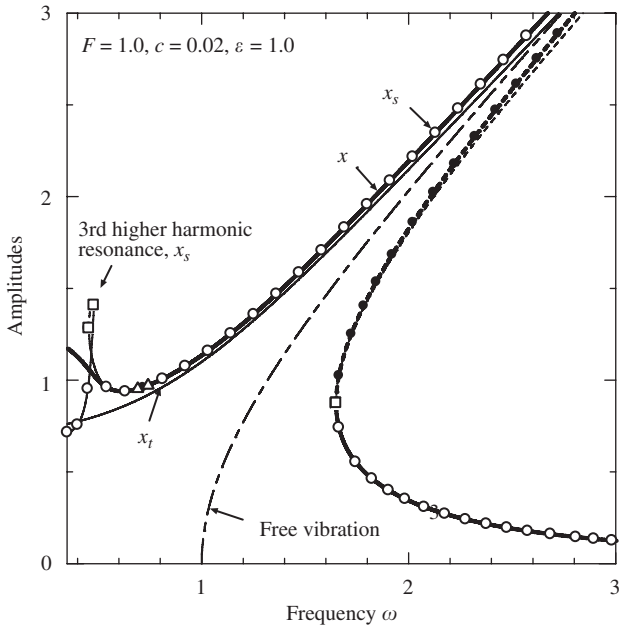

Fig. 1. Amplitude frequency response of a Duffing oscillator with hardening spring for $F = 1.0$, $c = 0.02$ and $\varepsilon = 1.0$. Averaging method of cn type: ▬ (stable solution) and ▪▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and - - - - -(unstable solution). Shooting method: —○— (stable solution) and - - -●- - - (unstable solution). Exact solution of undamped free vibration: – · – · . □ and △ denote boundary between stable and unstable regions for saddle-node and pitchfork bifurcations, respectively.

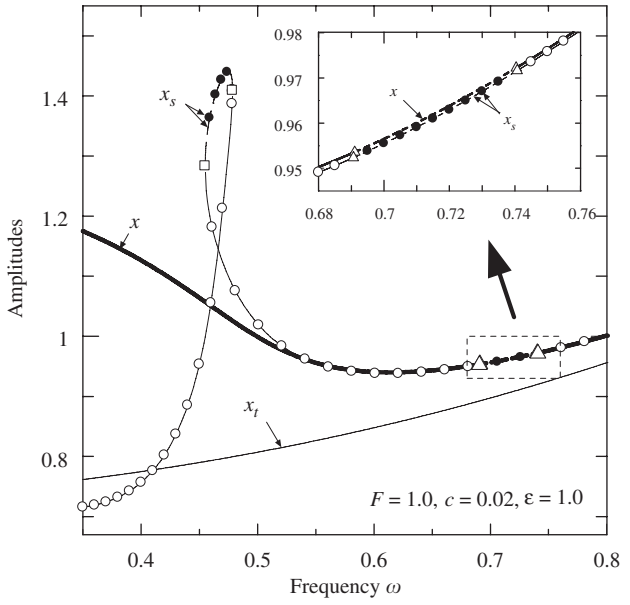

Fig. 2. Detailed view of the amplitude frequency response of a Duffing oscillator with hardening spring in the low frequency region ($\omega = 0.35$–0.8). Averaging method of cn type: ▬ (stable solution) and ▪▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and - - - - -(unstable solution). Shooting method: —○— (stable solution) and - - -●- - - (unstable solution). □ and △ denote boundary between stable and unstable regions for saddle-node and pitchfork bifurcations, respectively.

a highly accurate approximate solution, the method I also becomes highly accurate. As a result, method I is recommended for subsequent stability analyses in order to take advantage of the highly accurate solutions obtained by the proposed method.

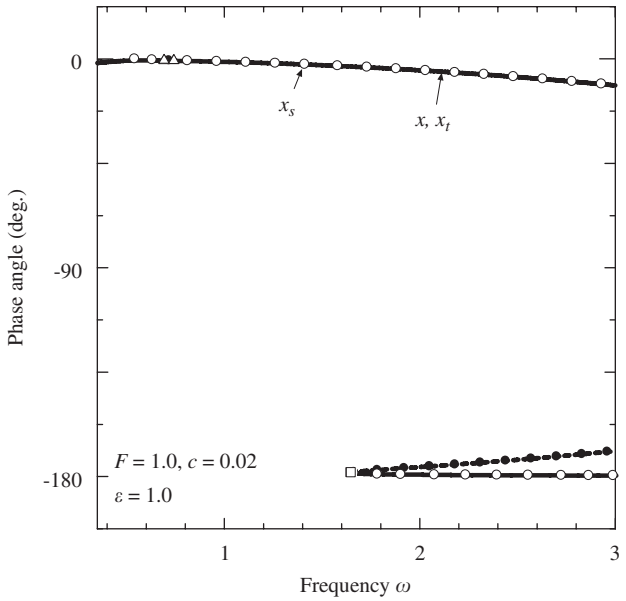

Fig. 3. Phase angle frequency response of a Duffing oscillator with hardening spring for $F = 1.0$, $c = 0.02$ and $\varepsilon = 1.0$. Averaging method of cn type: ▬ (stable solution) and ▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and - - - - - (unstable solution). Shooting method: —○— (stable solution) and - - -●- - - (unstable solution). Exact solution of undamped free vibration: – · – ·. □ and △ denote boundary between stable and unstable regions for saddle-node and pitchfork bifurcations, respectively.

Fig. 2 depicts the higher harmonic resonance of third-order calculated by the shooting method. Neither the proposed method nor the conventional method is capable of calculating the higher harmonic resonance of third-order.

Fig. 3 shows the frequency response of the phase angle. The phase angles obtained by the proposed method agree well with those determined by shooting method.

Next, the accuracy of the approximate solution x obtained by the averaging method of cn type is quantitatively verified by comparing with the numerical solution x_s obtained by the shooting method. The proposed method is favorable because it is capable of calculating the higher harmonic components of the solution x. Therefore, the accuracy is estimated by the relative error of the maximum amplitude $E_{\max}$ and the root mean square error E_{rms} defined as

$$E_{\max} = \left| \frac{A - x_{s,\max}}{x_{s,\max}} \right|, \tag{50}$$

$$E_{\mathrm{rms}} = \left[\frac{1}{2\pi} \int_0^{2\pi} (x - x_s)^2 \mathrm{d}u \right]^{1/2} = \sqrt{\frac{1}{2}} \sqrt{\sum_{n=1,3,5,\ldots} \{(a_n - a_{s,n})^2 + (b_n - b_{s,n})^2\}}, \tag{51}$$

where $x_{s,\max}$ denote the maximum amplitudes of x_s, $a_{s,n}$ and $b_{s,n}$, which are the nth-order Fourier cosine and sine coefficients of x_s that are calculated using an FFT. These nth-order Fourier cosine and sine coefficients of x are calculated using Fourier series equations of the Jacobian elliptic functions as follows:

$$\left.\begin{aligned} x(t) &= \sum_{n=1,3,5,\ldots} A_n \cos n(\omega t + \bar{\theta}) = \sum_{n=1,3,5,\ldots} [a_n \cos n\omega t + b_n \sin n\omega t], \\ A_n &= \frac{2\pi \bar{A}}{\bar{k}\bar{K}} \frac{\bar{q}^{n/2}}{1 + \bar{q}^n}, \quad a_n = A_n \cos n\bar{\theta}, \quad b_n = A_n \sin n\bar{\theta}, \quad \bar{q} = \exp\left(-\frac{\bar{L}}{\bar{K}}\right), \quad \bar{L} = K(\bar{l}). \end{aligned}\right\} \tag{52}$$

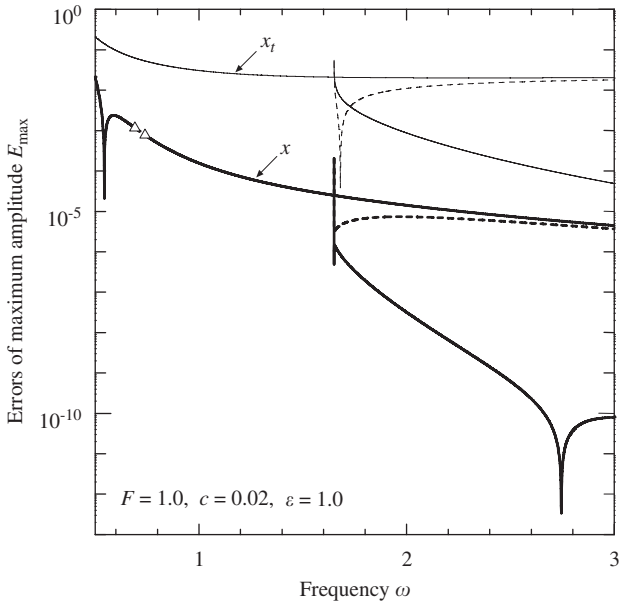

Fig. 4. Relative error of maximum amplitude E_{max} for a Duffing oscillator with hardening spring for $F = 1.0$, $c = 0.02$ and $\varepsilon = 1.0$. Averaging method of cn type: ▬▬ (stable solution) and ▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and - - - - - (unstable solution). △ denotes boundary between stable and unstable regions for pitchfork bifurcation.

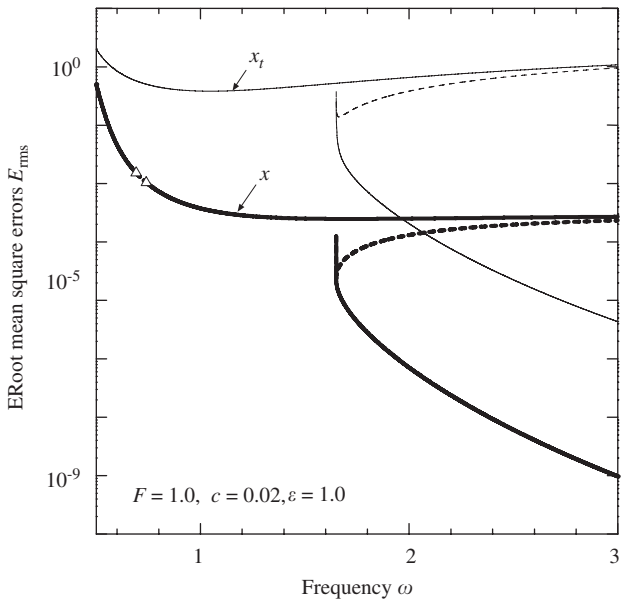

Fig. 5. Root mean square error E_{rms} for a Duffing oscillator with hardening spring for $F = 1.0$, $c = 0.02$ and $\varepsilon = 1.0$. Averaging method of cn type: ▬▬ (stable solution) and ▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and - - - - - (unstable solution). △ denotes boundary between stable and unstable regions for pitchfork bifurcation.

Figs. 4 and 5 display the relative error E_{max} and the root mean square error E_{rms}. The similar errors E_{max} and E_{rms} calculated for the conventional averaging method results are represented by the thin line in Figs. 4 and 5. These figures reveal that the accuracy of the proposed method's approximate solution is considerably improved over that for the conventional averaging method. In one region of Fig. 4, the response curve

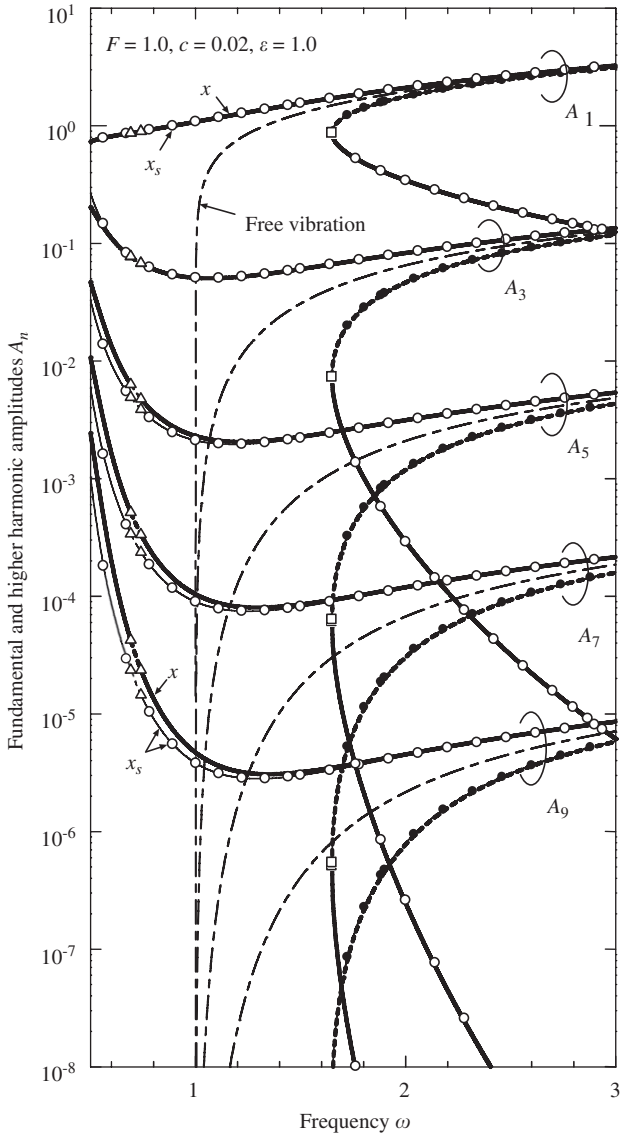

Fig. 6. Frequency response of the fundamental and higher harmonic amplitudes of a Duffing oscillator with hardening spring for $F = 1.0$, $c = 0.02$ and $\varepsilon = 1.0$. Averaging method of cn type: ▬▬ (stable solution) and ▪▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and - - - - - (unstable solution). Shooting method: —○— (stable solution) and - - -●- - - (unstable solution). Exact solution of undamped free vibration: – · – ·. □ and △ denote boundary between stable and unstable regions for saddle-node and pitchfork bifurcations, respectively.

resembles a sharp valley. At this point the maximum amplitudes calculated by the proposed method and the shooting method coincide with each other and $E_{\max}$ becomes zero. However, since the E_{rms} value for this point is not zero, the correct solution has not been obtained.

Fig. 6 shows the frequency response curves for the fundamental amplitude and the higher harmonic amplitudes of odd order from the third to ninth components. Both the fundamental and the higher harmonic amplitudes obtained by the proposed method coincide well with those obtained by the shooting method.

The waveform and the phase portrait for the stable solutions on the left flank of the main resonance ($\omega = 0.6$) are shown in Fig. 7. The conventional averaging method waveform is clearly different from the shooting method waveform due to the large deformation caused by higher harmonic components. On the other hand, both the maximum amplitude and the delicate distortion of the waveform obtained by proposed

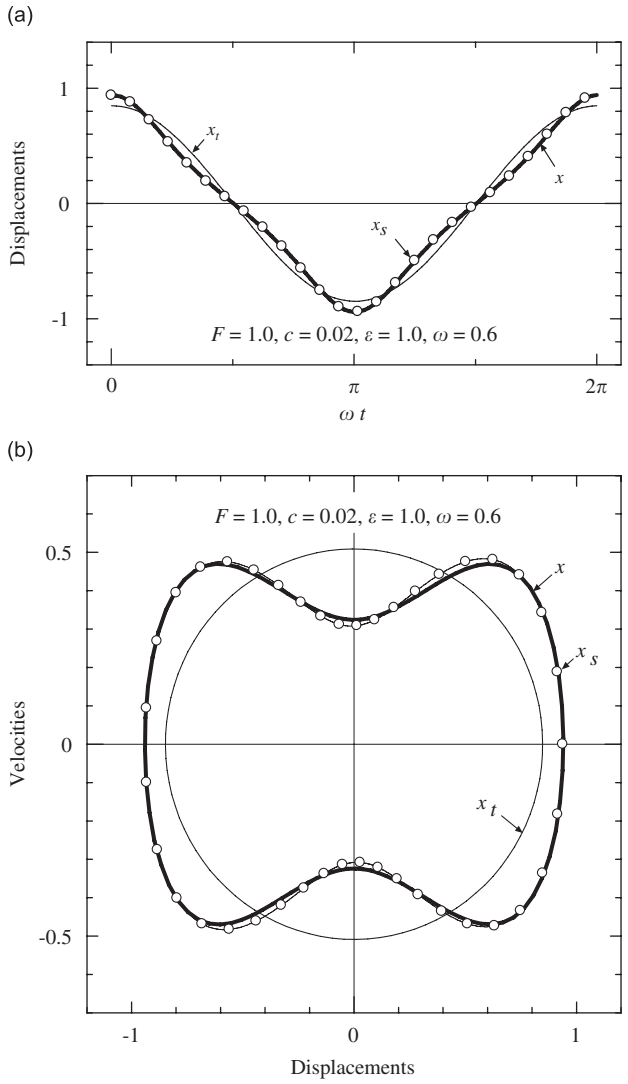

Fig. 7. Waveforms and phase portraits of the stable solution of a Duffing oscillator with a hardening spring for $F = 1.0$, $c = 0.02$ and $\varepsilon = 1.0$. Angular frequency of external force is $\omega = 0.6$: (a) waveforms and (b) phase portraits. Averaging method of cn type: ▬. Conventional averaging method: ——. Shooting method: —○—.

method agree with the numerical solution to a high degree of accuracy. This can be attributed to the method's ability to provide good approximations for higher harmonic components.

Fig. 8 reveals the influence of the magnitude of ε on the root mean square error E_{rms} for the stable solution at $\omega = 1.5$. The accuracy of the solution obtained by the proposed method improves rightfully as the magnitude of ε decreases.

Next, the calculation results for the proposed method and Roy's method [11] are compared for the case of a Duffing oscillator with hardening spring, $\beta_1 = \beta_3 = 0$ in Eq. (3). Roy's method is an averaging method that transforms the original governing equation into two first-order autonomous equations governing the energy and the phase variable. The form of the chosen transformation is given by unperturbed conservative orbits of the system. The approximate solution of Roy's method for a Duffing oscillator with hardening spring is assumed by following the Jacobian elliptic function:

$$x \equiv x(e,u) = A\,\mathrm{cn}(au,k), \quad y \equiv y(e,u) = \dot{x} = -Aa\,\mathrm{sn}(au,k)\mathrm{dn}(au,k), \tag{53}$$

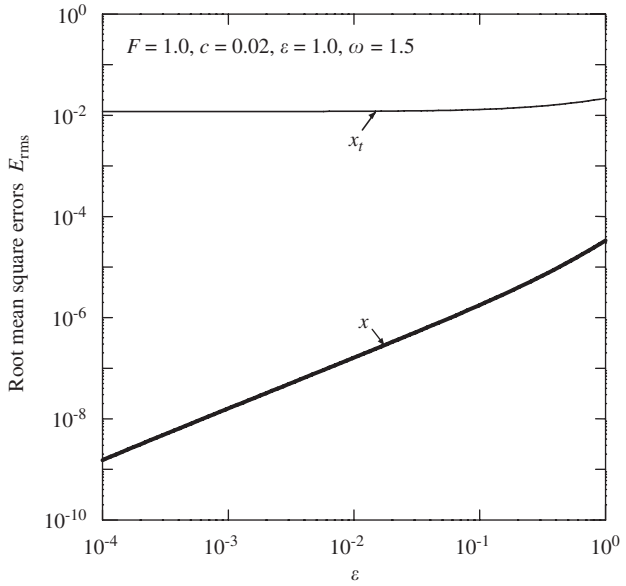

Fig. 8. Influence of the magnitude of ε on the root mean square error E_{rms} of stable solution for a Duffing oscillator with a hardening spring for $F=1.0$, $c=0.02$, $\varepsilon=1.0$ and $\omega=1.5$. Averaging method of cn type: ▬. Conventional averaging method: ——.

where $u=t+$constant. The relationship between amplitude A, modulus k, constant a and energy level e is

$$e=\frac{1}{2}A^2+\frac{1}{4}A^4,\quad a^2=A^2+1,\quad k^2=\frac{A^2}{2(1+A^2)}. \tag{54}$$

Above relations are derived by the condition of an exact solution of undamped free vibration of a Duffing oscillator with hardening spring. The equations for determining the unknown variables of Roy's method is

$$\left.\begin{aligned} cAa\rho_0+\frac{F}{2}\alpha_1\sin\theta=0,\\ \frac{\pi a}{2K}-\omega-\varepsilon\frac{\pi aF}{4K}\frac{\mathrm{d}A}{\mathrm{d}e}\left\{c_1+\frac{2k^2}{A^2l^2}(w_1-k^2v_1)\right\}\cos\theta=0,\end{aligned}\right\} \tag{55}$$

where $\mathrm{d}A/\mathrm{d}e=1/(A+A^3)$, θ is phase angle and α_1, c_1, v_1, w_1 and ρ_0 are the function of modulus k. ρ_0 is the average value of the function $\mathrm{sn}^2\,au\,\mathrm{dn}^2\,au$, α_1, c_1, v_1 and w_1 are the first Fourier coefficients of the functions $\mathrm{sn}\,au\,\mathrm{dn}\,au$, $\mathrm{cn}\,au$, $\mathrm{sn}^2\,au\,\mathrm{cn}\,au$ and $\mathrm{Z}(au)\mathrm{sn}\,au\,\mathrm{dn}\,au$, respectively. The analytical expressions of variables ρ_0, α_1, c_1, v_1 and w_1 are formulated in Ref. [11]. While solving Eqs. (54) and (55) with respect to A, θ, a and k using a numerical method, these unknown variables are determined and the steady-state solution can be obtained. Refer to Roy's paper in Ref. [11] for the detailed description of the computational theory.

The amplitude frequency response and the root mean square error defined by Eq. (51) for the approximate solutions of the proposed method and Roy's method are shown in Figs. 9 and 10. The parameters are set to $F=0.11$, $c=0.1$, $\varepsilon=1.0$. The computational result of Roy's method is depicted by the dash-dot line and indicated by x_R. Roy's method can calculate the approximate solution with high accuracy in the vicinity of the resonance peak, but the accuracy of the approximate solution for other frequency ranges is very poor. The approximate solution obtained by the proposed method agrees with the shooting method's numerical solution.

Roy's method uses the Jacobian elliptic function that is the exact solution of undamped free vibration for $\varepsilon=0$ in Eq. (3). For the case where the Jacobian elliptic function is used, modulus k is an important quantity that characterizes the approximate solution. Therefore, the optimum value for the modulus must be used, and this is the most difficult and important problem. It is believed that the decrease in computational accuracy of

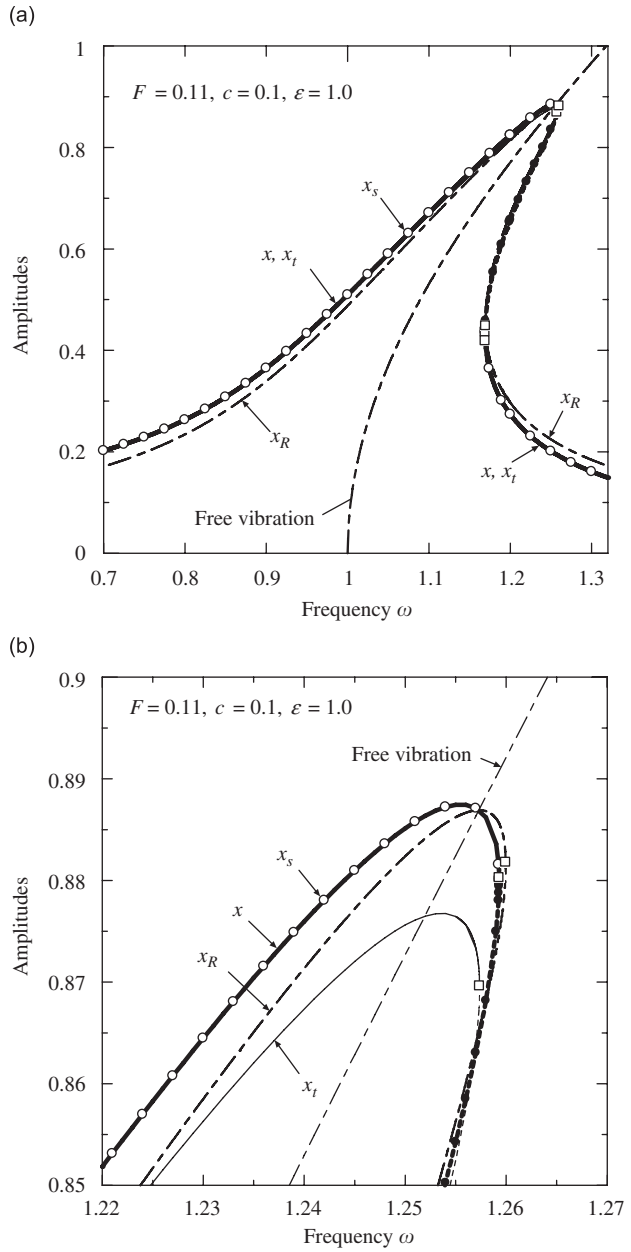

Fig. 9. Amplitude frequency response of a Duffing oscillator with hardening spring for $F = 0.11$, $c = 0.1$ and $\varepsilon = 1.0$: (a) $0.7 \leqslant \omega \leqslant 1.32$ and (b) detailed view of resonance peak ($1.22 \leqslant \omega \leqslant 1.27$). Averaging method of cn type: ▬▬ and ▪▪▪▪. Conventional averaging method: —— and - - - - -. Shooting method: —○— and - - -●- - -. Roy's method: — · —. Exact solution of undamped free vibration: – – – – –. □ denotes boundary between stable and unstable regions for saddle-node bifurcations.

Roy's method is caused by using Eq. (54) that holds only for the case of undamped free vibration. On the other hand, the proposed method does not use the relationship among amplitude, frequency and modulus of undamped free vibration shown in Eq. (11). By making use of the pseudo-averaged equation presented in Section 3, the proposed method can obtain the highly accurate approximate solution for a strongly nonlinear oscillator defined by Eq. (3) over a wide frequency range.

Finally, the computational speed of the proposed method is compared with that of the shooting method. A precise comparison of computational speed is difficult, because the calculation procedures for the two

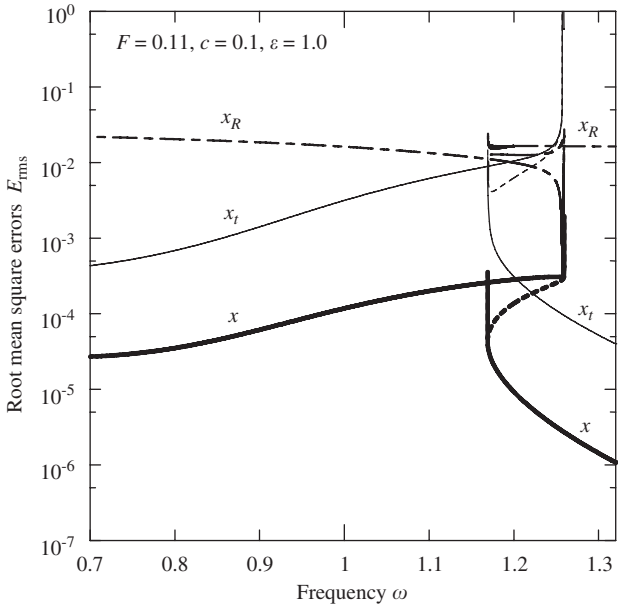

Fig. 10. Root mean square error E_{rms} for a Duffing oscillator with hardening spring for $F = 0.11$, $c = 0.1$ and $\varepsilon = 1.0$. Averaging method of cn type: ▬▬ and ▪▪▪▪. Conventional averaging method: —— and ------. Roy's method: — · —.

methods are fundamentally different. Therefore, the computational speeds of the two methods are approximately estimated.

Both the proposed method and the shooting method must use the Newton–Raphson method to obtain the solution. In order to roughly estimate the computational speed, the CPU time required for calculating the iteration process of the Newton–Raphson method was measured. In this process, the numerical integration based on Simpson's rule was used for computing $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$ in proposed method, and the RKG method was applied to solve the fundamental and variational equations in the shooting method simultaneously. It is well known that CPU times for numerical integration and the RKG method depend on the step-size $h = 2\pi/N$, where N is the number of partitions for one period 2π. Therefore, the above-mentioned CPU times $T_J{}^N$ of the proposed method and $T_S{}^N$ of the shooting method were measured for $N = 64, 128, \ldots, 2048$. In the measurements of $T_J{}^N$ and $T_S{}^N$, the system parameters were set to $F = 1.0$, $c = 0.02$, $\omega = 1.0$ and $\varepsilon = 1.0$.

Fig. 11 shows the CPU time ratio defined by

$$C_J^N = \frac{T_J^N}{T_J^{64}}, \quad C_S^N = \frac{T_S^N}{T_J^{64}}. \tag{56}$$

The symbols ■ and ○ denote $C_J{}^N$ and $C_S{}^N$, respectively. Comparing $C_J{}^N$ and $C_S{}^N$, the computational speed of the proposed method seems to be about 3 or 4 times faster than that of the shooting method for the same N. A similar property is confirmed for alternate system parameters and for other types of nonlinear oscillators. Therefore, it might be concluded that the proposed method is also superior to the shooting method from the viewpoint of computational efficiency.

5.3. Averaging method of cn type for Duffing–van der Pol oscillator with hardening spring

This example computes the periodic solution for a forced Duffing–van der Pol oscillator with hardening spring ($\beta_1 = \beta_3 = 1$) using the averaging method of cn type. The equation of van der Pol type was employed to model the self-excited oscillation circuit [16]. The perturbed function $f(\omega t, x, \dot{x})$ on the right-hand side of

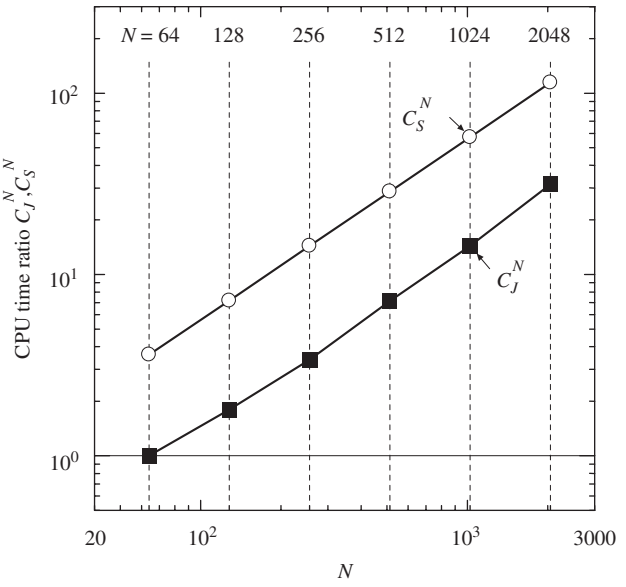

Fig. 11. Comparison of computation time. Averaging method of cn type: ■. Shooting method: ○.

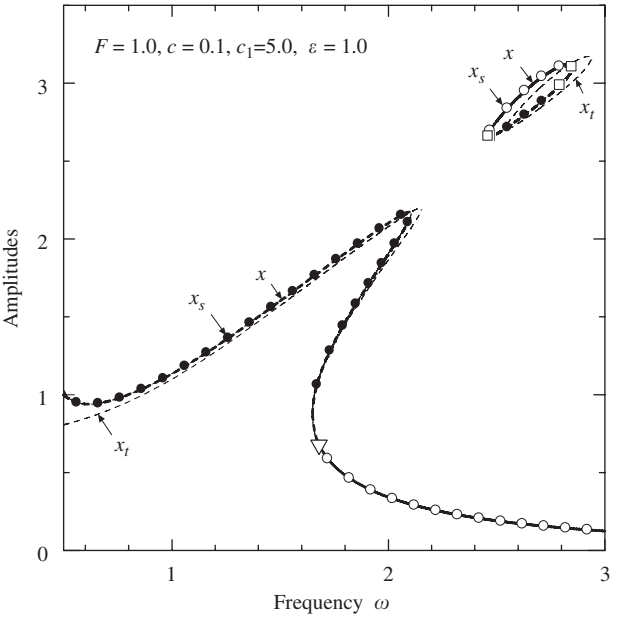

Fig. 12. Amplitude frequency response of a Duffing–van der Pol oscillator with hardening spring [Eq. (56)] for $F = 1.0$, $c = 0.1$, $c_1 = 5.0$ and $\varepsilon = 1.0$. Averaging method of cn type: ▬▬ (stable solution) and ▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and ------(unstable solution). Shooting method: —○— (stable solution) and ---●--- (unstable solution). □ and ▽ denote boundary between stable and unstable regions for saddle-node and Hopf bifurcations, respectively.

Eq. (3) is defined as

$$f(\omega t, x, \dot{x}) = F \cos \omega t - c(1 - c_1 x^2 + x^4)\dot{x}, \tag{57}$$

where F and ω are the amplitude and the angular frequency of external force, and c and c_1 are the damping coefficients of the van der Pol oscillator. For this example, the parameters have been set to $F = 1.0$, $c = 0.1$, $c_1 = 5.0$ and $\varepsilon = 1.0$.

Fig. 12 displays the frequency response of the amplitude and clearly shows that the approximate solution obtained by the proposed method coincides well with the numerical solution obtained by the shooting method. The Hopf bifurcation point appears at $\omega = 1.688$ on right flank of main resonance for both methods. In addition, the approximate solution calculated by the proposed method is more accurate than the solution obtained by the conventional averaging method.

5.4. Averaging method of sn type for pendulum oscillator

Next, the averaging method of sn type is applied to a pendulum oscillator. The dimensionless equation of motion of a forced pendulum oscillator with viscous damping is generally expressed as

$$\ddot{\phi} + \varepsilon c\dot{\phi} + \sin\,\phi = \varepsilon T\,\sin\,\omega t, \tag{58}$$

where ϕ is the pendulum's rotation angle, T and ω are the amplitude and the angular frequency of the harmonically excited torque, and c is the damping coefficient. Eq. (58) can be transformed by scaling variables ϕ and T according to $\phi = \sqrt{6}x$ and $T = \sqrt{6}F$. Using this technique, the governing equation becomes

$$\ddot{x} + x - x^3 = \varepsilon f(\omega t, x, \dot{x}), \tag{59}$$

where the perturbed function $f(\omega t, x, \dot{x})$ is defined by

$$f(\omega t, x, \dot{x}) = F\,\sin\,\omega t - c\dot{x} - \frac{1}{\varepsilon}\left[\sum_{n=3}^{\infty}\frac{(-6)^{n-1}}{(2n-1)!}x^{2n-1}\right] \tag{60}$$

and $\beta_1 = 1$, $\beta_3 = -1$ in Eq. (3).

The frequency response of the amplitude is shown in Fig. 13. For this example, the parameters are set to $F = 0.5/\sqrt{6} = 0.204\ldots(T = 0.5)$, $c = 0.317$ and $\varepsilon = 1.0$. The figure indicates that the frequency response curves obtained by the shooting method separate into upper-left and lower-right branches. The upper-left branch has a peninsular-like feature hanging down from the upper-left region of the skeleton curve. The

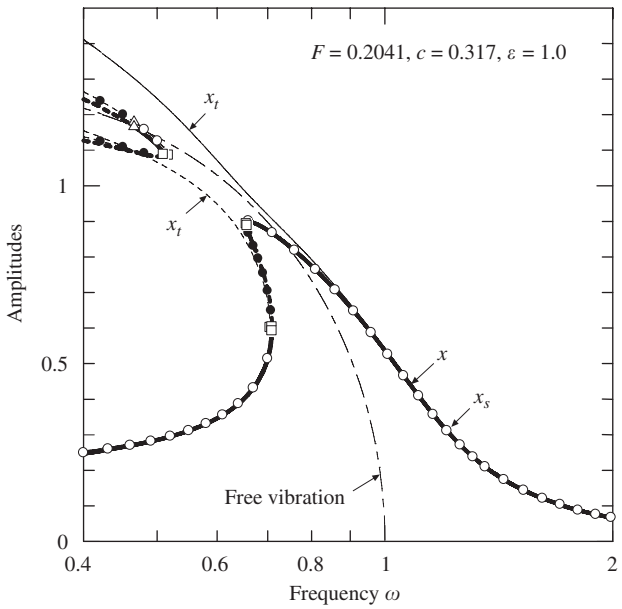

Fig. 13. Amplitude frequency response of a pendulum oscillator for $F = 0.204$, $c = 0.317$ and $\varepsilon = 1.0$. Averaging method of sn type: ▬ (stable solution) and ▪▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and - - - - - -(unstable solution). Shooting method: —○— (stable solution) and - - -●- - - (unstable solution). Exact solution of undamped free vibration — · —. □ and △ denote boundary between stable and unstable regions for saddle-node and pitchfork bifurcations, respectively.

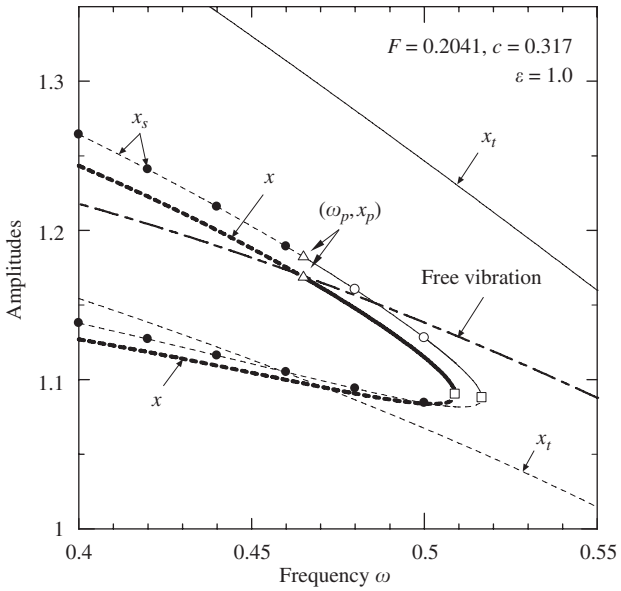

Fig. 14. Detailed view of the amplitude frequency response of the upper-left branch a pendulum oscillator containing a peninsular-like shape ($\omega = 0.4$–0.55). Averaging method of sn type: ▬▬ (stable solution) and ▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and ------(unstable solution). Shooting method: —○— (stable solution) and ---●--- (unstable solution). Exact solution of undamped free vibration — · —. □ and △ denote boundary between stable and unstable regions for saddle-node and pitchfork bifurcations, respectively.

lower-right branch takes on the well-known primary resonance form. The result obtained by the averaging method of sn type generates these two branches and corresponds with the solution obtained by the shooting method. On the other hand, the conventional averaging method generates two resonance branches that sandwich the skeleton curve. The shapes of the conventional method's frequency response curves for the parameter values used in the computation are qualitatively and quantitatively different from those generated by the shooting method.

Fig. 14 shows a detailed view of the upper-left frequency response curve. When the method I is used for the stability analysis, the pitchfork bifurcation point for the proposed method appears at $(\omega_p, x_p) = (0.4658, 1.168)$ in the upper branch. The pitchfork bifurcation point calculated by the shooting method is located at $(\omega_p, x_p) = (0.4652, 1.182)$. These numerical results are nearly coincident. When the averaging equation discussed in Section 4.2 is used as the stability criterion, all of the approximate solutions for the upper region of the upper-left branch take on a peninsular-like form and become stable solutions. In this case, the method II cannot determine the stability of the secondary unstable region as well as the result of Section 5.2.

5.5. Averaging method of dn type for a Duffing oscillator with snap-through spring and piecewise linear spring containing a dead zone

This final example calculates the computational results for a Duffing oscillator with a snap-through spring ($\beta_1 = -1$, $\beta_3 = 1$) that contains a piecewise linear spring with a dead zone. The perturbed function $f(\omega t, x, \dot{x})$ for this example is defined as

$$\left.\begin{aligned} f(\omega t, x, \dot{x}) &= F\cos\omega t - c\dot{x}, \quad (d_2 < x < d_1),\\ f(\omega t, x, \dot{x}) &= F\cos\omega t - c\dot{x} - k_1 x, \quad (x > d_1),\\ f(\omega t, x, \dot{x}) &= F\cos\omega t - c\dot{x} - k_2 x, \quad (x < d_2). \end{aligned}\right\} \tag{61}$$

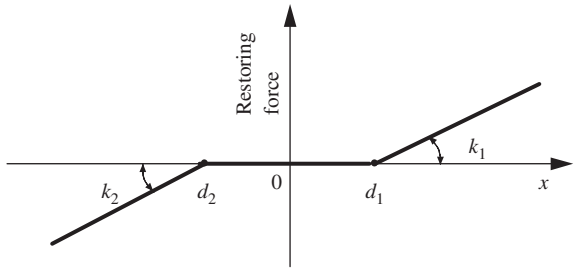

Fig. 15. Piecewise linear spring containing a dead zone.

The periodic solution of the half-swing mode vibrating around the one of the stable equilibrium points ($x = 1$) is determined using the averaging method of dn type.

Fig. 15 illustrates the restoring force for a piecewise linear spring containing a dead zone. The parameters are set to $F = 0.1$, $c = 0.15$ and $\varepsilon = 1.0$. The amplitude frequency response curves are shown in Fig. 16. Fig. 16(a) displays the frequency response curve for a normal Duffing oscillator with a snap-through spring system without a piecewise linear spring ($k_1 = k_2 = 0$) and Fig. 16(b) displays that for the same oscillator with a piecewise linear spring containing a dead zone ($k_1 = k_2 = 0.1$, $d_1 = 1.2$, $d_2 = 0.8$). Since the steady-state solution of these oscillators is a non-odd order solution and the waveform becomes asymmetrical, the vertical axes of Figs. 16(a) and (b) represent the peak-to-peak amplitude. The generating solution for the half-swing mode using the conventional averaging method can be expressed as

$$x_t = R + A_t \cos(\omega t + \theta_t), \quad \dot{x}_t = -A_t\omega \sin(\omega t + \theta_t), \tag{62}$$

where $A_t = A_t(t)$, $\theta_t = \theta_t(t)$ and R is a constant. The shapes of the frequency response curves for the conventional averaging method are both qualitatively and quantitatively different from those generated by the shooting method. In addition, the averaging method of dn type calculates the approximate solution with higher accuracy than the conventional averaging method.

6. Conclusions

An averaging method of elliptic type using the Jacobian elliptic functions is developed as a generating solution to obtain highly accurate steady-state solutions for strongly nonlinear oscillators with a single degree of freedom. This method is classified into the averaging method of cn, sn or dn types according to the Jacobian elliptic function used for the generating solution. The particular method is selected based upon the type of basic oscillator used, which is determined by β_1 and β_3 in Eq. (3). The proposed method can be widely applied to nonlinear oscillators having nonlinear restoring forces in the first- and third-order terms with respect to the displacement. The function $\Psi_i (i = 2-4,\ 8-10)$ in Eqs. (34) and (35) is used to determine unknown variables $\bar{A}$, $\bar{\theta}$ and $\bar{k}$, which are independent of the perturbed function $f(\omega t, x, \dot{x})$. These equations can be derived analytically, as shown in Eqs. (31)–(33). On the other hand, $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$ in Eqs. (34) and (35) are defined by each system and change depending on the form of the perturbed function $f(\omega t, x, \dot{x})$. In addition, it is difficult to analytically calculate $\bar{H}_1$, $\bar{H}_2$ and $\bar{H}_3$ for an arbitrary $f(\omega t, x, \dot{x})$. Therefore, Simpson's rule can be employed for the necessary numerical integration, allowing the proposed method to be applied to many oscillators.

Two types of stability analysis for the approximate solution obtained by the proposed method are also discussed. One is the method by using the variational equation of the fundamental equation and the other by using the variational equation of the averaging equation. Though the former is very accurate, the latter is unable to determine the secondary unstable region caused by a pitchfork and Hopf bifurcation point. The accuracy of the latter stability analysis is poor even when the approximate solution is obtained with a high degree of accuracy by the proposed method.

Four examples containing different oscillators were conducted. When the results of the proposed method were compared to the very accurate numerical solutions calculated by the shooting method, it was confirmed that the proposed method provides more accurate solutions than those obtained by the conventional

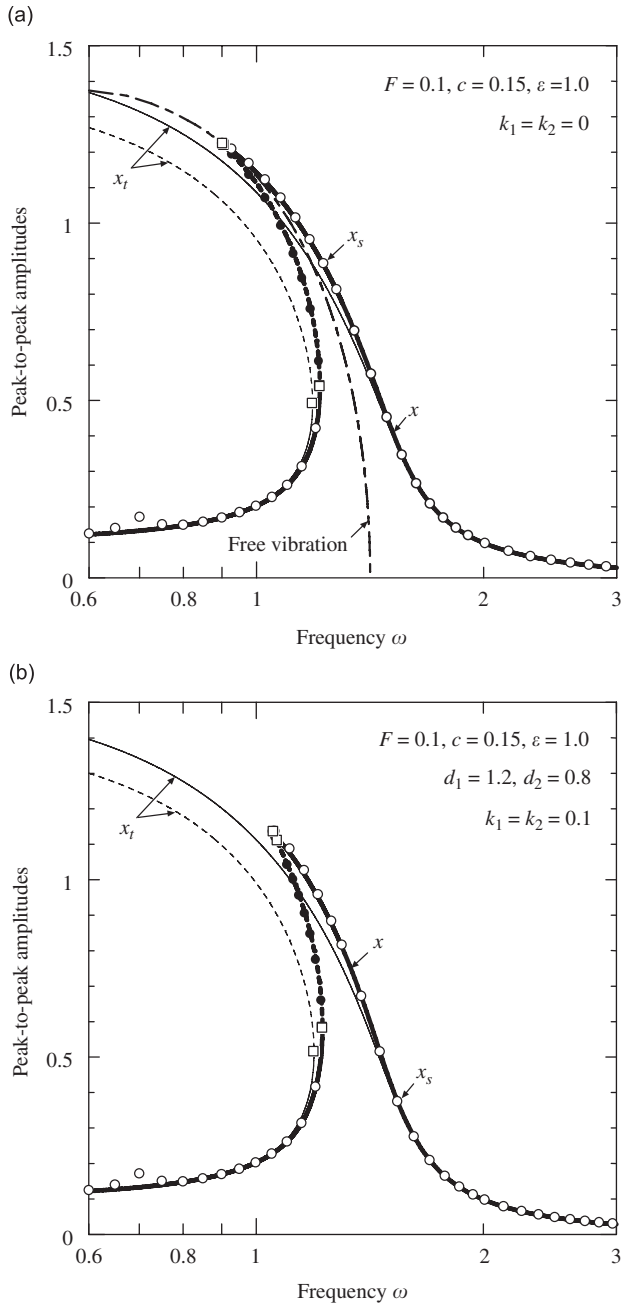

Fig. 16. Peak-to-peak frequency response of the half-swing mode of a Duffing oscillator with a snap-through spring for $F = 0.1$, $c = 0.15$ and $\varepsilon = 1.0$. (a) Normal Duffing oscillator with a snap-through spring. (b) Duffing oscillator with a snap-through spring and a piecewise linear spring containing a dead zone. Averaging method of dn type: ▬▬ (stable solution) and ▪▪▪▪ (unstable solution). Conventional averaging method: —— (stable solution) and - - - - - -(unstable solution). Shooting method: —○— (stable solution) and - - -●- - - (unstable solution). Exact solution of undamped free vibration — · — · . □ denotes boundary between stable and unstable regions for saddle-node bifurcation.

averaging method that uses trigonometric functions as a generating solution. Moreover, all of the numerical results agree well with those obtained by the shooting method for moderately large parameter values of the perturbed function. The averaging method of elliptic type also obtains similar computational accuracy values

for other oscillators. These results confirm that the proposed method is very effective in analyzing the steady-state solution of strongly nonlinear oscillators based on the Duffing equation.

Acknowledgment

The authors gratefully acknowledge the support of Grants-in-Aid for the Scientific Research in 2007 from the Japan Society for the Promotion of Science.

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